Reduction of Goldbach Failure Modes in the Divisibility Lattice

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Author: Sxo Cra

1. Preliminaries and Definitions

Definition 1 (Divisibility Lattice)

Let

$L = \{(x, y) \text{ in } N \times N\text{: y divides x }\}$.

This structure records all divisibility relations between positive integers.

Define the indicator relation

$B(x, y) = 1$ if and only if y divides x,

with x, y in N and $y \geq 2$.

Definition 2 (Prime and Composite)

An integer $n \geq 2$ is prime if and only if there does not exist an integer d with

$2 \leq d \leq n - 1$ such that d divides n.

An integer $n \geq 2$ is composite if and only if it is not prime.

Equivalently, n is composite if and only if there exists a prime q such that q divides n.

Definition 3 (Goldbach Predicate)

Let $E \geq 4$ be an even integer.

Define the Goldbach predicate G(E) as follows:

G(E) holds if and only if there exists an integer p with

$2 < p < E-2$

such that p is prime and E-p is prime.

2. Exact Characterization of Goldbach Failure

Theorem 1 (Exact Failure Characterization)

For any even integer $E \geq 4$, the following are equivalent:

1. Goldbach fails for E (that is, G(E) does not hold).

2. For all integers x with $2 \le x \le E - 2$, at least one of x or E - x is composite.
3. For all integers x with $2 \le x \le E - 2$, there exists a prime q such that

q divides x or q divides E - x.

Proof

($1 \Rightarrow 2$)

If Goldbach fails for E, then there does not exist any x in [2, E- 2] such that both x and E - x are prime. Therefore, for every such x, at least one of x or E - x must be composite.

($2 \Rightarrow 3$)

If x or E-x is composite, then by definition it has at least one prime divisor. Hence there exists a prime q dividing x or dividing E-x.

($3 \Rightarrow 1$)

If for every x in [2, E - 2] at least one of x or E - x has a prime divisor, then no pair (x, E − **x**) consists of two primes. Therefore Goldbach fails for E.

This completes the proof.

3. Mirror-Blocking Formulation Inside the Divisibility Lattice

Definition 4 (Prime Columns)

For each integer $x \geq 2$, the column x is the set of points (x, y) in L with $y \geq 2$.

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Column x is prime if it contains no points in L with $y \geq 2$.

Column x is composite if it contains at least one such point.

Definition 5 (Mirror Blocking for an Even Integer)

Fix an even integer $E \geq 4$.

A mirror-blocking configuration for E is a collection of divisibility points in L such that for every x in the interval $[2, E - 2]$, at least one of the two columns x or E-x is intersected by a point of the form (x, y) or $(\mathbf{E} - x, y)$ with $y \geq 2$.

Corollary 1 (Mirror-Blocking Equivalence)

Goldbach fails for E if and only if the interval of columns

$\{2, 3, \ldots, E-2\}$

is fully mirror-blocked in the divisibility lattice.

This is a direct restatement of Theorem 1 in lattice terms.

4. Allowed Operations on the Lattice

The analysis is restricted to the following operations, and no others:

1. Restriction: finite truncation in x or y
2. Translation: x mapped to x + c
3. Reflection: x mapped to C-x
4. Superposition: finite union of transformed copies
5. Scaling: (x, y) mapped to (k x, k y) for k in N

These operations preserve divisibility structure and introduce no new generators.

5. Blocking Layers

Definition 6 (Blocking Layer)

A blocking layer is a set HCL such that there exists a finite Y with the property that, for all sufficiently large x in a specified interval, the column x intersects H at some y ≤ Y.

Blocking layers represent bounded-depth obstructions that eliminate entire ranges of columns.

6. Closure Theorem

Theorem 2 (Closure Under Allowed Operations)

Let L be the divisibility lattice.

No finite composition of restriction, translation, reflection, superposition, or scaling can generate

a new bounded-depth blocking layer unless such a layer already exists in L itself.

Proof

All points of L lie on rays of the form $x = k\,y$, for fixed k in N.

The following properties are invariant under all allowed operations:

1. Depth monotonicity

For any ray $x = k\,y$, the depth y at which the ray intersects a given column is bounded by x/k. No allowed operation increases this bound.

2. Orientation rigidity

Rays have fixed rational slopes. Translation, reflection, and scaling preserve or reverse slopes but do not create new ones.

3. Finite intersection behavior

Each ray intersects a given column only finitely many times within bounded depth.

4. Additivity without generation

Superposition increases the number of rays but does not alter their individual depth bounds or orientations.

Therefore, any bounded-depth blocking layer obtained by allowed operations must already be present in the original lattice.

7. Application to Goldbach

Theorem 3 (Goldbach Reduction)

Within the divisibility lattice and the allowed operations:

If Goldbach fails for an even integer E, then there must exist a bounded-depth, mirror-symmetric blocking layer across the interval [2, E – 2].

By Theorem 2, no such blocking layer can be generated unless it already exists in the lattice.